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MORE REMARKS ON THE AFFINE SPACE PARTITION OF THE VARIETY OF N–STABLE FLAGS

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This brief note is a supplement to the announcement I submitted to the proceeding of a conference in algebraic combinatorics, 1990 RIMS, Kyoto, Japan [T1]. In the course of generalizing the interpretation of the polynomial $\sum_{\lambda\vdash n} \widetilde{K}_{\lambda(1^n)}(q)\widetilde{K}_{\lambda(1^n)}(t)$ to that of

$$\sum_{\lambda\vdash n} \widetilde{K}_{\lambda\mu}(q)\widetilde{K}_{\lambda(1^n)}(t) \tag{1}$$

(for any partition μ of n), we clarified that the partition of the variety of N-stable flags (where N is a fixed nilpotent linear transformation) first due to N. Spaltenstein ([Sp]) can be directly related to the partition of the variety of all flags into Schubert cells. In this note we add some more remarks on this partition. In particular, we will remark that a partition of some other varieties, called the Spaltenstein varieties, can also be related to the Schubert cells.

Let us recall the interpretation of the polynomial (1) with a slight generalization from that of [T1]. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_l)$ be a composition of n such that $\sigma_1, \sigma_2, \ldots, \sigma_l$, when arranged in the decreasing order, would give the partition μ. Let $\mathcal{T}_\sigma$ denote the set of row-decreasing tableaux of shape σ in which each symbol in the range 1 through n appear once (see Fig. 1).

5	3		
2			
10	7	4	1
9	8	6	

Fig. 1. An example of a row-decreasing tableau of shape $(2, 1, 4, 3)$

Now we define two statistics $l(T)$ and $\iota(T)$ on the set $\mathcal{T}_\sigma$. $l(T)$ is defined to be the sum of $l^{(i)}(T)$ for $1 \le i \le n-1$, where $l^{(i)}(T)$ is the number of entries of T greater than i in the area designated by the shade in Fig. 2 (determined by the position of i in T).

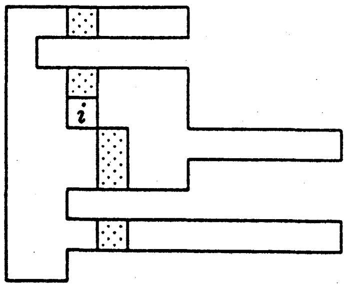

FIG. 2. THE AREA DETERMINED BY i

To define $\iota(T)$, we regard the boxes in the diagram of σ as secretly numbered from 1 to n as in Fig. 3.

(7)	(4)		
(8)			
(9)	(5)	(2)	(1)
(10)	(6)	(3)	

Fig. 3. hidden labels of boxes

Let us call these numbers the *hidden labels* of the boxes. Then $\iota(T)$ is equal to the sum of i (in the range 1 through $n-1$) such that, in T, the number $i+1$ lies in a box having a smaller hidden label that that of the box containing i. With these definitions, the polynomial (1) has the following expression as the generating function of these two statistics:

$$\sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(q)\tilde{K}_{\lambda(1^n)}(t) = \sum_{T \in \mathcal{T}_\sigma} q^{l(T)} t^{\iota(T)}.$$

It is easy to see that, for $\mu = (1^n)$, $\mathcal{T}$ can be identified with $\mathfrak{S}_n$ via the correspondence $w \leftrightarrow \begin{matrix} w^{-1}(1) \\ w^{-1}(2) \\ \vdots \\ w^{-1}(n) \end{matrix}$ and through this correspondence $l(T)$ reduces to the usual notion of the number of inversions of permutations, and $\iota(T)$ to the greater index.

Actually, $l(T)$ has some geometric meaning as follows. First let $\mathcal{B}$ denote the variety of all complete flags in $\mathbb{C}^n$. As is well known, $\mathcal{B}$ has a cell decomposition into locally closed subsets which are isomorphic to complex affine spaces of various dimensions. The cells are called the Schubert cells, and are exactly parametrized by the elements of $\mathfrak{S}_n$. The dimension of the cell labelled by $w \in \mathfrak{S}_n$ is equal to

$l(w)$, the number of inversions in w:

$$\mathcal{B} = \coprod_{w \in \mathfrak{S}_n} X_w, \quad X_w \approx \mathbb{C}^{l(w)}.$$

For a composition σ of n, which is a rearrangement of the partition μ, we define a nilpotent transformation N_σ on $\mathbb{C}^n$, making use of the hidden labelling of the boxes in the diagram of σ, as illustrated in Fig. 4.

$$N = N_\sigma : \begin{cases} e_7 \mapsto e_4 \mapsto 0 \\ e_8 \mapsto 0 \\ e_9 \mapsto e_5 \mapsto e_2 \mapsto e_1 \mapsto 0 \\ e_{10} \mapsto e_6 \mapsto e_3 \mapsto 0 \end{cases}$$

Fig. 4. N_σ for $\sigma = (2,1,4,3)$ (see also Fig. 3)

N_σ is conjugate to the Jordan canonical form with cells of sizes $\mu_1, \mu_2, \ldots$. Let $\mathcal{B}_{N_\sigma}$ denote the subvariety of $\mathcal{B}$ consisting of flags stable under N_σ (a flag $(V_1, V_2, \ldots, V_n)$ is called stable under N_σ if each of its components V_i is N_σ-stable). Then the intersection of the Schubert cell X_w with $\mathcal{B}_{N_\sigma}$ is not empty if and only if the tableau of shape σ obtained by filling the box i (hidden labelling) with $w^{-1}(i)$ (call it T) is row-decreasing — i.e. an element of $\mathcal{T}_\sigma$. Moreover, if the intersection is nonempty, then it is isomorphic to a complex affine space of dimension $l(T)$ in the above sense. Therefore we obtain a partition of $\mathcal{B}_{N_\sigma}$ into locally closed subspaces which are isomorphic to affine spaces of various dimensions:

$$\mathcal{B}_{N_\sigma} = \coprod_{T \in \mathcal{T}_\sigma} X_T, \quad X_T \approx \mathbb{C}^{l(T)}, \quad X_T = X_{w_T} \cap \mathcal{B}_{N_\sigma},$$

where w_T is a permutation such that $w_T(i)$ gives the hidden label of the box in T containing the number i.

This partition is a special case of the one given in [Sh]. Our point here is that such a partition is realized by intersecting with the Schubert cells if we choose the nilpotent transformation appropriately. Note that this choice of N_σ is slightly more general than that in [T1] where we only considered the case $\sigma = \mu$. We should be careful that this is not a cell decomposition in general, in the sense that the closure of an affine piece is not a union of some lower dimensional pieces; even in a simple case where $n = 3$ and $\sigma = \mu = (2,1)$.

Next we consider the variety of flags with jumps in dimensions. Let J be any subset of $\{1, 2, \ldots, n-1\}$ (the set of jumps). Let $\mathcal{P}^J$ denote the variety of flags with jumps at J, namely the set of chains (with respect to inclusion) of linear subspaces of $\mathbb{C}^n$ whose dimensions does *not* belong to J:

$$\mathcal{P}^J = \{ (V_i)_{i \in [1,n] \setminus J} \mid i_1, i_2 \in [1,n] \setminus J,\ i_1 < i_2 \implies V_{i_1} \subset V_{i_2} \}.$$

$\mathcal{P}^J$ also has a cell decomposition into Schubert cells Y_w^J, where w runs over the permutations in $\mathfrak{S}_n$ satisfying $w(j) < w(j+1)$ for $j \in J$, i.e. the set $\mathfrak{S}_n^J$ of minimal length coset representatives for $\mathfrak{S}_n/\mathfrak{S}_J$, where $\mathfrak{S}_J$ is the subgroup of $\mathfrak{S}_n$ generated by $\{ s_j = (j, j+1) \mid j \in J \}$. For these w, the Schubert cell X_w in the complete flag variety $\mathcal{B}$ is mapped isomorphically onto Y_w^J by the natural projection π^J of $\mathcal{B}$ onto $\mathcal{P}^J$:

$$\mathcal{P}^J = \coprod_{w \in \mathfrak{S}_n^J} Y_w^J, \quad w \in \mathfrak{S}_n^J \text{ implies } Y_w^J \xleftarrow{\sim} X_w \approx \mathbb{C}^{l(w)}.$$

Now let $\mathcal{P}^J_{N_\sigma}$ denote its N_σ-stable part. Then $\mathcal{P}^J_{N_\sigma}$ is decomposed as a finite union of affine spaces as follows, by intersecting with the Schubert cells:

$$\mathcal{P}^J_{N_\sigma} = \coprod_{T \in \mathcal{T}_\sigma^J} Y_T^J, \quad T \in \mathcal{T}_\sigma^J \text{ implies } Y_T^J \xleftarrow{\sim} X_T \approx \mathbb{C}^{l(T)}, \tag{2}$$

where $\mathcal{T}_\sigma^J$ is the set of tableaux $T \in \mathcal{T}_\sigma$ whose corresponding permutations w_T belong to the set of minimal length coset representatives $\mathfrak{S}_n^J$.

An affine space partition of this variety was first given by R. Hotta and N. Shimomura in [Sh] and [HSh]. The partition (2) seems to be just a dual of their partition, in the sense that, in (2) the *cotypes* of V_i for $(V_i) \in \mathcal{P}^J_{N_\sigma}$ (in other words the Jordan types of the nilpotent transformations induced by N_σ on the $\mathbb{C}^n/V_i$) are "constant" on each piece, whereas in [Sh] and [HSh] the *types* of V_i (the Jordan types of the nilpotent transformations induced by N_σ on the V_i) are constant on each piece. As a method of counting the Poincaré polynomial of the variety $\mathcal{P}^J_{N_\sigma}$, our method of counting $l(T)$ can easily shown to be equivalent to their method.

Now we look into another type of varieties called the Spaltenstein varieties. Let $(\mathcal{P}^J_{N_\sigma})^0$ denote the subvariety of $\mathcal{P}^J_{N_\sigma}$ consisting of jumping flags $(V_i)_{i\in[1,n]\setminus J}$ such that the transformation induced by N_σ on consecutive quotients $V_{i'}/V_i$ (where $i < i'$ are consecutive members in $[1,n] \setminus J$) are all zero. Another way to describe $(\mathcal{P}^J_{N_\sigma})^0$ is the variety of parabolic subgroups of $GL(n, \mathbb{C})$, conjugate to the standard one P_J (generated by the upper triangular Borel subgroup B and permutation matrices of simple reflections s_j, $j \in J$), and containing N_σ in the nilpotent radicals of their Lie algabras.

This time we put $\widetilde{\mathfrak{S}_n^J}$ to be the set of *maximal* length coset representatives, namely the permutations $w \in \mathfrak{S}_n$ such that $w(j) > w(j+1)$ for all $j \in J$. Putting $\widetilde{\mathcal{T}_\sigma^J}$ to be the subset of $\mathcal{T}_\sigma$ consisting of tableaux T whose corresponding words w_T belong to $\widetilde{\mathfrak{S}_n^J}$, we again have the following decomposition:

$$(\mathcal{P}^J_{N_\sigma})^0 = \coprod_{T \in \widetilde{\mathcal{T}_\sigma^J}} \widetilde{Y_T^J}, \quad \widetilde{Y_T^J} \approx \mathbb{C}^{l(T)-d_J} \longleftarrow X_T,$$

where the last arrow is not an isomorphism this time, but a (trivial) vector bundle with fibers of dimension $d_J = \sum_{k=1}^{m} \frac{1}{2}(i_k - i_{k-1})(i_k - i_{k-1} - 1)$, $i_1 < i_2 < \cdots < i_m$ being the elements of $[1,n] \setminus J$ arranged in the increasing order (i_m always being n), and i_0 denoting 0 for convenience. (This d_J is also equal to the maximum of the lengths of elements of $\mathfrak{S}_J$ and also to $\dim P_J - \dim B$.)

Example. $n = 4$, $\mu = \sigma = (2,1,1)$

The labels of the affine pieces

3-dim.	$\begin{array}{ll}4&3\\2&\\1&\end{array}$	$\begin{array}{ll}4&1\\3&\\2&\end{array}$	$\begin{array}{ll}4&2\\3&\\1&\end{array}$		
2-dim.	$\begin{array}{ll}4&3\\1&\\2&\end{array}$	$\begin{array}{ll}3&1\\4&\\2&\end{array}$	$\begin{array}{ll}3&2\\4&\\1&\end{array}$	$\begin{array}{ll}4&1\\2&\\3&\end{array}$	$\begin{array}{ll}4&2\\1&\\3&\end{array}$
1-dim.	$\begin{array}{ll}3&1\\2&\\4&\end{array}$	$\begin{array}{ll}3&2\\1&\\4&\end{array}$	$\begin{array}{ll}2&1\\4&\\3&\end{array}$		
0-dim.	$\begin{array}{ll}2&1\\3&\\4&\end{array}$				

For $J = \{3\}$, the tableaux inside the fence constitute the set $\mathcal{T}_\sigma^J$ and the remaining ones consisute $\widetilde{\mathcal{T}_\sigma^J}$. Considering that $d_J = 1$, we see that the Poincaré polynomial of $\mathcal{P}_{N_\sigma}^J$ is $q^3 + 3q^2 + 2q + 1$ and that of $\left(\mathcal{P}_{N_\sigma}^J\right)^0$ is $2q^2 + 2q + 1$.

Question. In general, let $\nu^{(k)}$ be a partition of $i_k - i_{k-1}$ for $k = 1, 2, \dots, m$. The set of flags (V_i) in $\mathcal{P}_{N_\sigma}^J$ such that N_σ induces a nilpotent of type $\nu^{(k)}$ on $V_{i_k}/V_{i_{k-1}}$ for all k does not have an affine space partition. What about the set of flags for which the transformation induced by N_σ belongs to the closure of type $\nu^{(k)}$? The case where all $\nu^{(k)}$ are single rows is $\mathcal{P}_{N_\sigma}^J$, and the case where all $\nu^{(k)}$ are single *columns* is $\left(\mathcal{P}_{N_\sigma}^J\right)^0$. The case where some of $\nu^{(k)}$ are single rows and the rest are single columns can be similarly treated. Some particular cases have also been tested.

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